# On Symmetrized Weight Compositions

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# First of All

• An **alphabet** $A$ is a finite left module over a finite ring $R$ with unity.

• A **code** of length **n** is just a submodule of $\mathbf{A^n}$. The **Hamming weight** counts the number of non-zero components in a tuple.

# Two Notions of Equivalence

Consider two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length $n$. We **may think** the two codes refer to **the same thing** in each of the following :

If $\mathcal{C}_1 \cong \mathcal{C}_2$ as (left) $R$-**submodules** of $A^n$ through an isomorphism that **preserves** Hamming weight (distance!),

**or**

if $\mathcal{C}_1$ and $\mathcal{C}_2$ are **monomially equivalent.**

# Is this true?

# Harvard, 1962

In her PhD thesis, **MacWilliams** proved the Hamming weight **EP** (later this was called being **MacWilliams**!) for **field alphabets.**

• The **alphabet** $A$ has the **Extension Property** (EP) with respect to Hamming weight if every **monomorphism** preserving Hamming weight extends to a **monomial transformation.**

# 1995

In [7], H.Ward and J.Wood **reproved** this via a **character theoretic** proof.

**Key Word:**
***generating characters***

**Now the question arises:**
**To what extent can this proof be generalized ?!**
**Can it work for arbitrary rings?**

**Nakayama's Definitions**

On Frobeniusean Algebras, 1939-1941

# Character Modules

(1) A finite ring $R$ is **Frobenius iff** ${}_R\widehat{R}$ is **cyclic.**

(2) $\mathrm{soc}(A)$ is **cyclic** if and only if $A$ can be **embedded into** ${}_R\widehat{R}$.

# Yes Frobenius is needed !!

**1-(Wood [8] 1999):** Every finite **Frobenius** ring has the extension property with respect to the Hamming weight.

Besides, Wood proved a partial converse (**for commutative rings**) in the same paper.

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**2-(Greferath, Nechaev, Wisbauer [3] 2004):** More generally, if $A$ is a Frobenius **bi-module** over the finite ring $R$, then $A$ has the extension property with respect to Hamming weight.

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**3-(Wood [10] 2009):** Wood reproved this same result following the style appearing in his 1999's paper.

**One more thing was proved...**

# Necessary and Sufficient

${}_{\mathbf{R}}\mathbf{A}$ is MacWilliams **if and only if**

1. $A$ is **pseudo-injective**, and
2. $A$ can be **embedded** in the character group $\widehat{\mathbf{R}}$ of $\mathbf{R}$ (or equivalently, $\mathbf{soc}(\mathbf{A})$ is **cyclic**).

## What Happens with Non-Cyclic Socles?

# One Year Earlier...

**Theorem:** Let $R = M_m(\mathbb{F}_q)$ be the ring of all $m \times m$ matrices over a finite field $\mathbb{F}_q$, and let $A = M_{m,k}(\mathbb{F}_q)$ be the left $R$-module of all $m \times k$ matrices over $\mathbb{F}_q$.
**If $\mathbf{k > m}$**, there exist linear codes $C_+, C_- \subset A^N, N = \prod_{i=1}^{k-1}(1+q^i)$, such that they are isomorphic through a weight preserving map **which does not** extend to a monomial transformation.

# Just Remember

that all this displayed so far concerns **Hamming** weight, so,

**Once again for *swc*?!**

• For any $G \preceq \mathrm{Aut}_R(A)$, define an **equivalence relation** $\sim$ on $A$: $a \sim b$ if $a = b\tau$ for some $\tau \in G$. Let $A/G$ denote the orbit space of this relation. The ***G-symmetrized weight composition*** is a function
**swc** : $A^n \times A/G \to \mathbb{Q}$ defined by,

$$\mathrm{swc}(x, a) = |\{i : x_i \sim a\}|,$$

where $x = (x_1, \ldots, x_n) \in A^n$ and $a \in A/G$. Thus, **swc** counts the number of components in each orbit.

# Analogies Deduced

• **In 2013**, in [2], **N. Elgarem, N. Megahed** and **J.Wood** proved that the **embeddability** in $\widehat{\mathbf{R}}$ (cyclic socle) is **sufficient** for satisfying the extension property with respect to the **G-symmetrized weight composition** for any subgroup $G$ of $\mathbf{Aut}_R(A)$,

but the **necessity** remained a **question.**

**A seemingly doomed trial suggests bridging to Hamming weight ...**

# Midway
# (Annihilator Weight)

**Define** an **equivalence relation** $\approx$ on $A$:

$$a \approx b \text{ if } Ann_a = Ann_b.$$

The ***Annihilator weight*** , denoted ***aw***, is then defined so that it counts the number of components in each orbit (i.e. having the same annihilator).

## Lemma

*In a **pseudo-injective** module, $\approx$ and $\sim_{\mathrm{Aut}_R(A)}$ make the **same** partition.*

### Theorem

*Let **R** be a **principal ideal ring**, $_{\mathbf{R}}\mathbf{A}$ a **pseudo-injective module**, and let **C** be a submodule of $\mathbf{A^n}$ for some n. Then a monomorphism $f : C \to A^n$ $(C \subseteq A^n)$ preserves **Hamming weight if and only if** it preserves $\mathrm{Aut}_R(A)$-**swc**.*

**Theorem**

*If ${}_{\mathbf{R}}\mathbf{A}$ is pseudo-injective, then $\mathbf{A}$ has the* ***extension property*** *with respect to $\mathrm{Aut}_R(A)$-**swc if and only if** $\mathrm{soc}(\mathbf{A})$ is **cyclic**.*

**Example:**

If $L$ is any finite field, and $K \subseteq L$ is a subfield. The $K$-module ${}_K L$ is pseudo-injective (by an extended basis argument). Thus the alphabet ${}_K L$ has the extension property with respect to $\mathrm{Aut}_K(L)$-**swc** if and only if $K = L$.

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# Thank You